Chapter 13

A Simple Characterization of Pessimism and Optimism: ε-Contamination Versus ε-Exuberance

13.1 Introduction and Summary

On Wall Street, there are bulls and bears among professional investors. On Main Street, ordinary people are sometimes overly optimistic about their future and at other times excessively pessimistic. Bulls and bears on Wall Street often have starkly different views about the markets even though available information is not so different among them. People on Main Street often switch from optimism to pessimism and *vice versa* quite easily even though there may not be noticeable change in their conditions.

The purpose of this chapter is to present a simple characterization of optimism and pessimism of this kind. We show that seemingly irrational, overly optimistic or excessively pessimistic behavior described in the previous paragraph can be in fact "rational" in the sense that it is consistent with axioms that are "reasonable," under fundamental uncertainty (that is, Knightian uncertainty) about the future.

The organization of this chapter is as follows. The remainder of this section explains main results and their implications in a non-technical manner. Section 2 presents mathematical preliminaries. Section 3 deals with pessimism and presents axioms and main results. The proof is contained in Section 4. Section 5 extends our analysis to the case of optimism, and presents axioms, main results and proofs.

13.1.1 Pessimism: ε-Contamination

Consider firstly pessimistic behavior. Suppose that (1) an economic agent is $(1-\varepsilon)\times 100\%$ certain that uncertainty she faces is characterized by a particular probability measure, but that (2) she has a fear that, with $\varepsilon \times 100\%$ chance, her opinion is completely wrong and she is left perfectly ignorant about the true measure in the future. In particular, there is always possibility of the worst case. This situation is often called "ε-contamination." That is, her confidence is partially contaminated by the fear of ignorance.

The ε-contamination is a special case of Knightian uncertainty or ambiguity in which the decision-maker faces not a single probability measure but a set of probability measures. Since it is analytically tractable, several authors have examined the ε-contamination or its variants in search behavior (Nishimura and Ozaki, 2001), portfolio choice (Chen and Epstein, 2002), learning (Nishimura and Ozaki, 2002) and voting (Chu and Liu, 2002). The ε-contamination comes up also in the statistics literature on robustness. See, for example, Berger (1985).

The first purpose of this chapter is to provide a simple set of behavioral axioms under which the decision-maker's preference is represented by the Choquet expected utility with the ε-contamination. These axioms are formal representation of the idea (1) and (2) described earlier.

13.1.2 Optimism: ε-Exuberance

Secondly, consider optimistic behavior. Suppose that (3) an economic agent is $(1-\varepsilon)\times 100\%$ certain that uncertainty she faces is characterized by a particular probability measure, but that (4) she is a little optimistic that, with $\varepsilon \times 100\%$ chance, her opinion is wrong and she comes across the best possible opportunity that she can imagine in the future, though she does not know exactly what is the best opportunity (or best probability measure). This situation can be called "ε-exuberance," following the famous quote of "irrational exuberance" by Alan Greenspan.[1] This economic agent has conviction that, however small, there is always a chance of a big success, although she is completely ignorant about the exact nature of this "big success."

[1] See Remarks by Chairman Alan Greenspan at the Annual Dinner and Francis Boyer Lecture of the American Enterprise Institute for Public Research, Washington, D. C., December 5, 1996.

The second purpose of this chapter is to provide a simple set of behavioral axioms under which the decision-maker's preference is represented by the maximax criterion and equivalently the Choquet expected utility with the ε-exuberance. These axioms are formal representation of the idea (3) and (4) described earlier.

In the ε-contamination, we hypothesize that a decision maker is not perfectly certain about the most likely probability measure and she thinks there is always possibility that the worst case happens. In the ε-exuberance, we assume the opposite: there is always possibility that the best case happens.

The ε-exuberance is an intuitive representation of optimistic behavior under Knightian uncertainty or ambiguity in which the decision-maker faces not a single probability measure but a set of probability measures. Optimistic behavior has recently been analyzed in insurance markets by Bracha and Brown (2010), who also explore its axiomatic foundation based on the framework of variational preferences developed by Maccheroni, Marinacci and Rustichini (2006). In contrast, we employ the lottery framework of Anscome and Auman (1963), which enables us to get intuitive interpretation of ε-exuberance.

13.1.3 Pessimism and Optimism: a Symmetry

In our framework, pessimism and optimism are symmetric. Investors on Wall Street almost agree on the most likely probability measure, with $(1-\varepsilon)\times 100\%$ certainty, which is based on information available to them. If there is no insider (private) information, their most likely probability measure is not so different. However, as mortal being, investors are different with their attitude toward Knightian uncertainty in the future. Some are pessimistic thinking of the worst case and others are optimistic hoping for the best outcome. However, there is no *a priori* rationale for choosing one against the other: both are equally plausible and acceptable.

A similar argument may hold for workers and consumers in Main Street. Facing Knightian uncertainty in the future, and having no convincing guidance, they may change from ε-contamination to ε-exuberance and *vice versa*. In retrospection, we observe such a flip-flop change in behavior in everyday life.

13.1.4 Example: Uniform Distribution

Let $\varepsilon \in (0,1)$ and let $a < b$. Suppose that an economic agent is $(1-\varepsilon)\times 100\%$ certain that uncertainty she faces is characterized by the uniform distribution over $[a, b]$ with its distribution function given by $F_0(x) = (x - a)/(b - a)$ for $x \in [a, b]$.

Now assume that her belief is represented by the ε-contamination of F_0. Then, the distribution function F is given by $F(x) = (1 - \varepsilon)F_0$ for $x \in [a, b)$ and $F(b) = 1$. Hence, the "density" function is given by

$$\chi_{[a,b)}(x)\frac{1-\varepsilon}{b-a}$$

where χ is the indicator function.

Next, assume that her belief is represented by the ε-exuberance of F_0. Then, the distribution function F is given by $F(a) = 0$ and $F(x) = (1 - \varepsilon)F_0 + \varepsilon$ for $x \in (a, b]$. Hence, the "density" function is given by

$$\chi_{(a,b]}(x)\frac{1-\varepsilon}{b-a} + \varepsilon\delta(x)$$

where δ is the *delta distribution* or *Dirac's delta* such that

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)$$

for any continuous function $f : \mathbb{R} \to \mathbb{R}$.

This specification is applicable whenever the primitive density function is continuous and makes it easy applying these concepts to empirical research.

13.2 Preliminaries

The definitions and notations in this paragraph is standard in the literature. The basic reference is Schmeidler (1989). Let (S, Σ) be a measurable space, where S is the set of states of the world and Σ is an algebra on it. Let Y be a mixture space. We call an element of Y a *lottery*. Given $y, y' \in Y$ and $\lambda \in [0, 1]$, we denote by $\lambda y + (1 - \lambda)y'$ the compound lottery. A *simple lottery act*, or more simply, an *act*, is a Y-valued Σ-measurable function on S whose range is a finite subset of Y. The set of acts is denoted by L_0. An act whose range is a singleton is referred to as a *constant act* and the set of constant acts is denoted by L_c. The decision-maker's preference is given by a binary relation $\succsim$ on L_0. A binary relation $\succsim$ is a *weak order* by definition if it is complete and transitive.[2] We define a binary relation over Y by

[2]Note that a binary relation $\succsim$ is a weak order if and only if $\succ$ is asymmetric and negatively transitive, where a binary relation $\succ$ is *asymmetric* if $(\forall f, g \in L_0)\ f \succ g \Rightarrow g \not\succ f$ and it is *negatively transitive* if $(\forall f, g, h \in L_0)\ [f \not\succ g$ and $g \not\succ h] \Rightarrow f \not\succ h$.

restricting $\succeq$ on L_c and denote it by the same symbol $\succeq$. We say that two acts, f and g, are *comonotonic* if $(\forall s, t \in S)\ [\, f(s) \succ f(t) \Rightarrow g(t) \not\succ g(s)\,]$.

In the following discussion, the "worst-limit" constant act as well as the "best-limit" one plays a crucial role. Given $f \in L_0$, let $Y_{\min f}$ be the subset of Y representing f's worst-limit costant act, defined by $Y_{\min f} = \{\, y \in Y \mid (\forall s)\ y \preceq f(s) \text{ and } (\exists s)\ y = f(s)\,\}$. Since f is a simple act, $Y_{\min f}$ is nonempty when $\succeq$ is a weak order. We henceforth denote by $y_{\min f}$ an arbitrary element of $Y_{\min f}$. Also, $Y_{\max f}$ and $y_{\max f}$ are defined symmetrically.

Given $f, g \in L_0$ and $\lambda \in [0,1]$, a compound lottery act $\lambda f + (1-\lambda) g$ or $f_\lambda g$ is defined by $(\forall s)\ (\lambda f + (1-\lambda) g)(s) = \lambda f(s) + (1-\lambda) g(s)$. By this operation, L_0 turns out to be a mixture space. A special case of the compound lottery act will turn to be important. We often consider $\lambda y_{\min f} + (1-\lambda) y_{\max f}$, that is, a compound act of the worst-limit act with probability λ and the best-limit act with probability $1-\lambda$.

13.3 Pessimistic Behavior: Axioms and Main Results

In addition to Schmeidler's (1989) well-known axioms[3]: (i) weak order; (ii) comonotonic independence; (iv) continuity; (v) monotonicity and (vii) nondegeneracy, we consider the following axioms which may be imposed on a binary relation $\succeq$ defined on L_0. In the axioms, f and g denote arbitrary elements in L_0 and λ denotes an arbitrary real number such that $\lambda \in (0,1]$.

The first axiom requires that any simple lottery act f is dominated by some compound lottery act of its worst-limit and best-limit constant acts. In the axiom, ε is a real number such that $\varepsilon \in [0,1)$. The axiom requires that the given relation should hold with respect to this ε. Therefore, whether the axiom is satisfied or not depends on ε, and hence, it is labeled (viii-ε), rather than (viii).

(viii-ε) (ε-Dominance)

$$(1-\varepsilon) y_{\max f} + \varepsilon y_{\min f} \succeq f.$$

Under (i), (ii), (iv), (v), (vii) and (viii-ε), it can be shown (see Lemma 1 of Section 4) that all $f \in L_0$ has the following ε-contamination equivalence:

$$(\forall f \in L_0)(\exists y_f \in L_c)\ \ f \sim (1-\varepsilon) y_f + \varepsilon y_{\min f}\,, \qquad (13.1)$$

[3] For explicit statements of these axioms, the readers are referred to Schmeidler (1989).

where ε is the one with which (viii-ε) holds. This property shows that all simple lottery acts have their own equivalent compound act consisting of its worst-limit constant act with probability ε and some constant act y_f with probability $1-\varepsilon$. Clearly, y_f defined in (13.6) is one way of representing f. We hereafter call it *f's equivalent constant act in ε-contamination equivalence*.

The next axiom concerns ordering among these equivalent constant acts in ε-contamination equivalence. In the axiom, ε is a real number such that $\varepsilon \in [0,1)$. By the same reason given for (viii-ε), we label it (ix-ε), rather than (ix).

(ix-ε) (Worst-limit irrelevance) Both of the following hold:

(ix-ε-1) (Affine irrelevance) If there exist y_f, y_g, $y_{f\lambda g} \in L_c$ such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$, $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$ and $f\lambda g \sim (1-\varepsilon)y_{f\lambda g} + \varepsilon y_{\min f\lambda g}$, then $y_{f\lambda g} \sim \lambda y_f + (1-\lambda)y_g$; and

(ix-ε-2) (Monotone irrelevance) If $(\forall s)$ $f(s) \succeq g(s)$ and there exist y_f, $y_g \in L_c$ such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$ and $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$, then $y_f \succeq y_g$.

Axiom (ix-ε-1) means that if y_f, y_g and $y_{f\lambda g}$ are the equivalent acts of f, g and $f\lambda g$ in ε-contamination equivalence, respectively, then $y_{f\lambda g} \sim \lambda y_f + (1-\lambda)y_g$, regardless of characteristics of the worst-limits $y_{\min f}$, $y_{\min g}$ and $y_{\min f\lambda g}$. Similarly, Axiom (ix-ε-2) implies that if $f(s) \succeq g(s)$ for all s, then $y_f \succeq y_g$, regardless of characteristics of the worst-limits $y_{\min f}$ and $y_{\min g}$. These two axioms imply that the worst limits are irrelevant in ordering among equivalent constant acts in ε-contamination equivalence.

Axioms (viii-ε) and (ix-ε) are closely related to the axioms of Anscombe and Aumann (1963), especially their independence axiom (or equivalently, Schmeidler's (iii) independence). In fact they can be considered as a natural extension of the Anscombe-Aumann theory to the case in which the decision-maker has a fear of the worst outcome with the possibility of ε all the time. We will turn to this issue in the next section.

As shown in Schmeidler (1989), the five axioms (i), (ii), (iv), (v) and (vii) as a whole characterize the preference which is represented by the Choquet expected utility with respect to some capacity.[4] The main results of this paper are the following theorem and corollary. The proof is relegated to Section 4.

[4] For a related axiomatization, see Gilboa and Schmeidler (1989).

Theorem 33 *Given any $\varepsilon \in [0,1)$, a binary relation $\succeq$ defined on L_0 satisfies (i), (ii), (iv), (v), (vii), (viii-ε) and (ix-ε) if and only if there exist a unique finitely additive probability measure μ on (S,Σ), an affine function $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation, such that*

$$f \succeq g \Leftrightarrow (1-\varepsilon)\int_S u(f(s))\, d\mu(s) + \varepsilon \min_{s\in S} u(f(s)) \geq (1-\varepsilon)\int_S u(g(s))\, d\mu(s) + \varepsilon \min_{s\in S} u(g(s))\,.$$

Let $\mathcal{M} = \mathcal{M}(S,\Sigma)$ be the set of finitely additive probability measures (probability charges) on (S,Σ), let $\varepsilon \in [0,1)$, and let $\mu \in \mathcal{M}$. Let us now define *ε-contamination of* μ, $\{\mu\}^\varepsilon$, which is a subset of $\mathcal{M}$, by $\{\mu\}^\varepsilon = \{\, (1-\varepsilon)\mu + \varepsilon q \mid q \in \mathcal{M}\,\}$.[5] Then, it follows that

$$\begin{aligned}(\forall f \in L_0) \quad \int_S u(f(s))\, d\{\mu\}^\varepsilon(s) &\equiv \min\left\{ \int_S u(f(s))\, dp(s) \,\middle|\, p \in \{\mu\}^\varepsilon \right\} \\ &= (1-\varepsilon)\int_S u(f(s))\, d\mu(s) + \varepsilon \min_{s\in S} u(f(s))\,.\end{aligned}$$

Therefore, the following corollary is immediate.

Corollary 9 *Given any $\varepsilon \in [0,1)$, a binary relation $\succeq$ defined on L_0 satisfies (i), (ii), (iv), (v), (vii), (viii-ε) and (ix-ε) if and only if there exist a unique finitely additive probability measure μ on (S,Σ), an affine function $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation, such that*

$$f \succeq g \Leftrightarrow \int_S u(f(s))\, d\{\mu\}^\varepsilon(s) \geq \int_S u(g(s))\, d\{\mu\}^\varepsilon(s)\,.$$

13.4 Pessimistic Behavior: Proof

The necessity of the axioms in Theorem 1 can be easily verified. We prove the sufficiency of them in this section.

13.4.1 A Lemma and the Anscombe-Aumann Theory

This subsection proves a lemma and then shows that our set of axioms can be considered as an extension of the Anscombe-Aumann theory to the case

[5] The capacity ν corresponding to ε-contamination of μ is given by

$$(\forall A \in \Sigma) \quad \nu(A) = \begin{cases} (1-\varepsilon)\mu(A) & \text{if} \quad A \neq S \\ 1 & \text{if} \quad A = S\,. \end{cases}$$

where the decision-maker considers the possibility of the worst outcome with the possibility of ε all the time (ε-contamination).

Lemma 21 *Let $\varepsilon \in [0,1)$ and assume that (i), (ii), (iv), (v), (vii) and (viii-ε) hold. Then,*

$$(\forall f \in L_0)(\exists y_f \in L_c) \quad f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}\,.$$

Proof. Let $f \in L_0$. Then, $y^* \equiv (1-\varepsilon)y_{\max f} + \varepsilon y_{\min f} \succeq f \succeq y_{\min f} \equiv y_*$, where the first and second orderings hold true by (viii-ε) and by (v), respectively. In the rest of proof, we assume that $y^* \succ f \succ y_*$ since the lemma would follow immediately otherwise. Then, since any pair of constant acts is comonotonic, a similar argument to Kreps (1988, pp.47-48) shows that $f \sim a^* y^* + (1-a^*)y_*$, where a^* is defined by $a^* = \sup\{\, a \in [0,1] \mid f \succeq ay^* + (1-a)y_* \,\}$. On the other hand,

$$\begin{aligned} a^* y^* + (1-a^*)y_* &= a^*((1-\varepsilon)y_{\max f} + \varepsilon y_{\min f}) + (1-a^*)((1-\varepsilon)y_{\min f} + \varepsilon y_{\min f}) \\ &= (1-\varepsilon)(a^* y_{\max f} + (1-a^*)y_{\min f}) + \varepsilon y_{\min f}\,. \end{aligned}$$

Therefore, to define $y_f = a^* y_{\max f} + (1-a^*)y_{\min f}$ completes the proof. ■

The next theorem is well-known.

Theorem 34 (Anscombe and Aumann, 1963) *A binary relation $\succ$ defined on L_0 satisfies (i), (iii) independence, (iv), (v) and (vii) if and only if there exist a unique finitely additive probability measure μ on (S, Σ) and an affine function $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation, such that*

$$f \succ g \;\Leftrightarrow\; \int_S u(f(s))\, d\mu(s) > \int_S u(g(s))\, d\mu(s)\,. \tag{13.2}$$

We now show that Axioms (i), (ii), (iv), (v), (vii) and (viii-0) and (ix-0), which are special cases of (viii-ε) and (ix-ε) by setting $\varepsilon = 0$, are necessary and sufficient for the Anscombe-Aumann axioms (i), (iii), (iv), (v) and (vii).

Proposition 14 *(i), (ii), (iv), (v), (vii), (viii-0) and (ix-0) $\Leftrightarrow$ (i), (iii), (iv), (v) and (vii).*

Proof. This paragraph proves that (i), (iii), (iv) and (v) $\Rightarrow$ (viii-0) and (ix-0). It is immediate that (v) implies (viii-0) and that (i) and (v) imply (ix-0-2). Then, we only need to prove that (ix-0-1) holds. This holds if we show the following claim:

$$(\forall f, g, h \in L_0) \quad f \sim g \;\Rightarrow\; \lambda f + (1-\lambda)h \sim \lambda g + (1-\lambda)h\,, \tag{13.3}$$

for it follows from (13.3) that if $f \sim y_f$, $g \sim y_g$ and $f_\lambda g \sim y_{f_\lambda g}$, then $\lambda y_f + (1-\lambda)y_g \sim \lambda y_f + (1-\lambda)g \sim \lambda f + (1-\lambda)g = f_\lambda g \sim y_{f_\lambda g}$. However, (i), (iii) and (iv) imply (13.3) by Kreps (1988, p.46, Lemma 5.6(c)).

This paragraph proves that (i), (ii), (iv), (v), (vii), (viii-0) and (ix-0) $\Rightarrow$ (iii). Lemma 1 proves that $(\forall f \in L_0)(\exists y_f \in L_c)\ f \sim y_f$ (simply let $\varepsilon = 0$ there). Let y_f, y_g, y_h, $y_{f_\lambda h}$, $y_{g_\lambda h} \in L_c$ be such that $f \sim y_f$, $g \sim y_g$, $h \sim y_h$, $f_\lambda h \sim y_{f_\lambda h}$ and $g_\lambda h \sim y_{g_\lambda h}$, and let $f \succ g$. Then, (i) implies that $y_f \succ y_g$. Since any pair of constant acts is comonotonic, (ii) implies that $\lambda y_f + (1-\lambda)y_h \succ \lambda y_g + (1-\lambda)y_h$. Finally, (i) and (viii-0-1) imply that $f_\lambda h \sim y_{f_\lambda h} \succ y_{g_\lambda h} \sim g_\lambda h$.

By combining the first two paragraphs, the proof is complete. ■

13.4.2 A Definition of $\succ^*$ and Some Preliminary Lemmas

We define a binary relation $\succ^*$ on L_0 induced by $\succ$ as follows:

$$f \succ^* g \;\Leftrightarrow\; [f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f} \;\text{ and }\; g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g} \;\Rightarrow\; y_f \succ y_g]\,,$$

where y_f and y_g are arbitrary elements of L_c. By definition, $f \succ^* g$ holds true whenever there does not exist such a y_f and/or y_g. Clearly, y_f [y_g] is, when it exists, f [g]'s equivalent constant act in ε-contamination equivalence. Thus, the binary relation $\succ^*$ is induced by the original preferences over these equivalent constant acts.

We define $\succeq^*$ and $\sim^*$ from $\succ^*$ by: $\succeq^* \Leftrightarrow \not\prec^*$ and $\sim^* \Leftrightarrow [\not\succ^* \text{ and } \not\prec^*]$. A binary relation on Y is naturally induced from $\succeq^*$ as its restriction on L_c and it is denoted by the same symbol, $\succeq^*$. Then, the following lemmas hold.

Lemma 22 *Assume that (i) holds, let $f \in L_0$ and let $y_f \in L_c$. If $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$, then $f \sim^* y_f$.*

Proof. Suppose that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$. It always holds that $y_f \sim (1-\varepsilon)y_f + \varepsilon y_{\min y_f}$ since $y_{\min y_f} = y_f$. Furthermore, $y_f \not\succ y_f$ since $\succ$ is asymmetric by (i). Therefore, by the definition of $\succ^*$, it follows that $f \not\succ^* y_f$. Similarly, $y_f \not\succ^* f$. Therefore, $f \sim^* y_f$. ■

Lemma 23 *Assume that (i) and (ii) hold. Then, $\succ$ and $\succ^*$ coincide on L_c.*

Proof. Let $y, y' \in Y$. First, assume that $y \succ^* y'$. Note that $y \sim (1-\varepsilon)y + \varepsilon y_{\min y}$ and $y' \sim (1-\varepsilon)y' + \varepsilon y'_{\min y'}$ hold since $(\forall y \in Y)\ y_{\min y} = y$. Hence, it follows from the definition of $\succ^*$ that $y \succ y'$. Second, assume that $y \succ y'$. Let $\bar{y}$ and $\bar{y}'$ be arbitrary constant acts such that (a) $y \sim (1-\varepsilon)\bar{y} + \varepsilon y_{\min y}$ and (b) $y' \sim (1-\varepsilon)\bar{y}' + \varepsilon y'_{\min y'}$. From (a), it holds that $(1-\varepsilon)y + \varepsilon y_{\min y} = y \sim (1-\varepsilon)\bar{y} + \varepsilon y_{\min y}$. Therefore, (i) and (ii) imply that $y \sim \bar{y}$ (recall that any pair of constant acts is comonotonic). Similarly, it holds from (b) that $y' \sim \bar{y}'$. Finally, (i) and the assumption that $y \succ y'$ show that $\bar{y} \succ \bar{y}'$, which in turn shows that $y \succ^* y'$ by the definition of $\succ^*$. ■

13.4.3 $\succ^*$ satisfies the Anscombe-Aumman Axioms

In this subsection, we show that the binary relation $\succeq^*$ satisfies the axioms postulated in Anscombe and Aumann (1963). Let $\varepsilon \in [0, 1)$. We henceforth suppress "-ε" and simply write as (viii) and (ix). Throughout this and the next subsections, we always assume that $\succeq$ satisfies (i), (ii), (iv), (v), (vii), (viii) and (ix).

Lemma 24 *The binary relation $\succeq^*$ satisfies (i), (iii), (iv), (v) and (vii).*

Proof. **(i) Weak Order.** We prove this by showing that $\succ^*$ is asymmetric and negatively transitive (see the footnote 2). (Asymmetry) Assume that $f \succ^* g$. Also suppose that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$ and that $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$. The existence of constant acts, y_f and y_g, is guaranteed by Lemma 1. Then, it follows from the definition of $\succ^*$ that $y_f \succ y_g$ and the asymmetry of $\succ$ implies that $y_g \not\succ y_f$. Hence, the definition of $\succ^*$ implies that $g \not\succ^* f$.

(Negative Transitivity) Assume that $f \not\succ^* g$ and $g \not\succ^* h$. Then, there exist constant acts y_f and y_g such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$, $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$ and $y_f \not\succ y_g$, and there exist constant acts y'_g and y_h such that $g \sim (1-\varepsilon)y'_g + \varepsilon y'_{\min g}$, $h \sim (1-\varepsilon)y_h + \varepsilon y_{\min h}$ and $y'_g \not\succ y_h$. It then holds that $(1-\varepsilon)y_g + \varepsilon y_{\min g} \sim g \sim (1-\varepsilon)y'_g + \varepsilon y'_{\min g} \sim (1-\varepsilon)y'_g + \varepsilon y_{\min g}$, where the last indifference relation holds since $y_{\min g} \sim y'_{\min g}$ and since (13.3) holds on L_c because $\succeq$ satisfies (i), (iii) and (iv) when restricted on L_c (see the proof of Proposition 1). Therefore, (i) and (ii) imply that $y_g \sim y'_g$ (recall that any pair of constant acts are comonotonic). Hence, (i) implies that $y_f \not\succ y_h$, which shows that $f \not\succ^* h$.

(iii) Independence. Assume that $f \succ^* g$ and let $y_{f_\lambda h}$ and $y_{g_\lambda h}$ be any constant acts such that $\lambda f + (1-\lambda)h \sim (1-\varepsilon)y_{f_\lambda h} + \varepsilon y_{\min f_\lambda h}$ and $\lambda g + (1-\lambda)h \sim (1-\varepsilon)y_{g_\lambda h} + \varepsilon y_{\min g_\lambda h}$. We show that $y_{f_\lambda h} \succ y_{g_\lambda h}$, which completes the proof by the definition of $\succ^*$. By Lemma 1 and the assumption that $f \succ^* g$, there exist constant acts y_f, y_g and y_h such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$, $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$, $h \sim (1-\varepsilon)y_h + \varepsilon y_{\min h}$ and $y_f \succ y_g$. Since any pair of constant acts is comonotonic, (ii) implies that $\lambda y_f + (1-\lambda)y_h \succ \lambda y_g + (1-\lambda)y_h$. On the other hand, (ix-1) implies that $\lambda y_f + (1-\lambda)y_h \sim y_{f_\lambda h}$ and $\lambda y_g + (1-\lambda)y_h \sim y_{g_\lambda h}$. Therefore, (i) shows that $y_{f_\lambda h} \succ y_{g_\lambda h}$.

(iv) Continuity. Assume that $f \succ^* g$ and $g \succ^* h$ and let y_f, y_g and y_h be any constant acts such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$, $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$ and $h \sim (1-\varepsilon)y_h + \varepsilon y_{\min h}$. Such y_f, y_g and y_h exist by Lemma 1. By the assumption that $f \succ^* g$ and $g \succ^* h$ and the definition of $\succ^*$, it follows that $y_f \succ y_g$ and $y_g \succ y_h$. Then, (iv) implies that there exists $\alpha \in (0,1)$ such that $\alpha y_f + (1-\alpha)y_h \succ y_g$. Let $y_{f_\alpha h}$ be any constant act such that $\alpha f + (1-\alpha)h \sim (1-\varepsilon)y_{f_\alpha h} + \varepsilon y_{\min f_\alpha h}$. Then, (ix-1) implies that $y_{f_\alpha h} \sim \alpha y_f + (1-\alpha)y_h$. Therefore, (i) shows that $y_{f_\alpha h} \succ y_g$, which in turn shows that $\alpha f + (1-\alpha)h \succ^* g$ by the definition of $\succ^*$. A similar proof applies for the existence of $\beta \in (0,1)$ such that $g \succ^* \beta f + (1-\beta)h$.

(v) Monotonicity. Suppose that $(\forall s \in S)\ f(s) \succeq^* g(s)$. Since $\succ^*$ and $\succ$ coincide on L_c (Lemma 3), it follows that $(\forall s \in S)\ f(s) \succeq g(s)$. Let y_f and y_g be constant acts such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$ and $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$. Such y_f and y_g exist by Lemma 1. Then, (ix-2) implies that $y_f \succeq y_g$, or equivalently, $y_g \not\succ y_f$. Therefore, it follows from the definition of $\succ^*$ that $g \not\succ^* f$, implying that $f \succeq^* g$.

(vii) Nondegeneracy. From (v) and (vii), it follows that $(\exists y, y' \in Y)\ y \succ y'$. Since $\succ^*$ and $\succ$ coincide on L_c (Lemma 3), $y \succ^* y'$. ∎

13.4.4 Completion of Proof

Anscombe and Aumann's theorem (1963) and Lemma 4 show that there exist a unique finitely additive probability measure μ on (S, Σ) and an affine function $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation, such that

$$f \succ^* g \;\Leftrightarrow\; \int_S u(f(s))\, d\mu(s) > \int_S u(g(s))\, d\mu(s)\,. \tag{13.4}$$

Define $J^* : L_0 \to \mathbb{R}$ by

$$(\forall f \in L_0) \quad J^*(f) = \int_S u(f(s))\, d\mu(s) \tag{13.5}$$

and define $J : L_0 \to \mathbb{R}$ by $(\forall f \in L_0)$ $J(f) = u((1-\varepsilon)y_f + \varepsilon y_{\min f})$, where $y_f \in L_c$ is f's equivalent constant act in ε-contamination equivalence whose existence is guaranteed by Lemma 1. Since u represents $\succ^*$ on L_c by (13.4) and $\succ^*$ and $\succ$ coincide on L_c by Lemma 3, u represents $\succ$ on L_c. Therefore, J is well-defined and represents $\succ$ on L_0. Finally, we have

$$\begin{aligned} J(f) &= u((1-\varepsilon)y_f + \varepsilon y_{\min f}) \ = \ (1-\varepsilon)u(y_f) + \varepsilon u(y_{\min f}) \\ &= (1-\varepsilon)J^*(y_f) + \varepsilon \min_s u(f(s)) \ = \ (1-\varepsilon)J^*(f) + \varepsilon \min_s u(f(s)) \\ &= (1-\varepsilon)\int_S u(f(s))\, d\mu(s) + \varepsilon \min_s u(f(s)), \end{aligned}$$

where the second equality holds by u's affinity; the third equality holds by the definition of J^* and the fact that u represents $\succ$ on Y; the fourth equality holds by Lemma 2 and the fact that J^* represents $\succ^*$; and the last equality holds by (13.5). Since J represents $\succ$ on L_0, the proof is complete. ■

13.5 Optimistic Behavior: Axioms, Main Results and Proof

13.5.1 Introduction

Suppose that an economic agent is $(1-\varepsilon) \times 100\%$ certain that uncertainty she faces is characterized by a particular probability measure, but that she is a little optimistic that, with $\varepsilon \times 100\%$ chance, her opinion is wrong and she comes across the best possible opportunity that she can imagine in the present as well as in the future, though she does not know exactly what is the best opportunity (or best probability measure). This situation can be called "ε-exuberance." This economic agent has conviction that, however small, there is always a chance of a big success, although she is completely ignorant about the exact nature of this "big success". By introspection, it seems not so uncommon that laymen may sometimes show this kind of small exuberance or "excessive optimism". Moreover, not-so-small-ε exuberance may be found in asset markets from time to time, as suggested by Alan Greenspan.[6]

The ε-exuberance is an intuitive representation of optimistic behavior under Knightian uncertainty or ambiguity in which the decision-maker faces

[6]See Remarks by Chairman Alan Greenspan at the Annual Dinner and Francis Boyer Lecture of the American Enterprise Institute for Public Research, Washington, D. C., December 5, 1996.

not a single probability measure but a set of probability measures. Optimistic behavior has recently been analyzed in insurance markets by Bracha and Brown (2010), who also explore its axiomatic foundation based on the framework of variational preferences developed by Maccheroni, Marinacci and Rustichini (2006). In contrast, we employ the lottery framework of Anscome and Auman (1963), which enables us to get intuitive interpretation of ε-exuberance.

The ε-exuberance also related (and exactly opposite) to the ε-contamination of confidence, where an economic agent is $(1-\varepsilon)\times 100\%$ certain that uncertainty she faces is characterized by a particular probability measure (her confidence), but that she has a fear that,with $\varepsilon \times 100\%$ chance, her conviction is completely wrong and she is left perfectly ignorant about the true measure. Since it is analytically tractable, several authors have examined the ε-contamination or its variants in search behavior (Nishimura and Ozaki, 2004), portfolio choice (Chen and Epstein, 2002), learning (Nishimura and Ozaki, 2002) and voting (Chu and Liu, 2002).[7]

The purpose of this paper is to provide a simple set of behavioral axioms under which the decision-maker's preference is represented by the maximax criterion and equivalently the Choquet expected utility with the ε-exuberance. This paper's representation of ε-exuberance is closely related to Nishimura and Ozaki's representation of ε-contamination (Nishimura and Ozaki 2006).

13.5.2 An Axiomatic Foundation

Preliminaries

The definitions and notations in this paragraph is standard in the literature. The basic reference is Schmeidler (1989). Let (S,Σ) be a measurable space, where S is the set of states of the world and Σ is an algebra on it. Let Y be a mixture space. We call an element of Y a *lottery*. Given $y, y' \in Y$ and $\lambda \in [0,1]$, we denote by $\lambda y + (1-\lambda)y'$ the compound lottery. A *simple lottery act*, or more simply, an *act*, is a Y-valued Σ-measurable function on S whose range is a finite subset of Y. The set of acts is denoted by L_0. An act whose range is a singleton is referred to as a *constant act* and the set of constant acts is denoted by L_c. The decision-maker's preference is given by a binary relation $\succeq$ on L_0. A binary relation $\succeq$ is a *weak order* by definition

[7] The ε-contamination comes up also in the statistics literature on robustness. See, for example, Berger (1985). Hansen and Sargent (2008) examine macroeconomic consequences when policy makers face model uncertainty and employ robust control.

if it is complete and transitive.[8] We define a binary relation over Y by restricting $\succeq$ on L_c and denote it by the same symbol $\succeq$. We say that two acts, f and g, are *comonotonic* if $(\forall s,t \in S)\ [\, f(s) \succ f(t) \Rightarrow g(t) \not\succ g(s)\,]$.

In the following discussion, the "worst-limit" constant act as well as the "best-limit" one plays a crucial role. Given $f \in L_0$, let $Y_{\min f}$ be the subset of Y representing f's worst-limit costant act, defined by $Y_{\min f} = \{\, y \in Y \,|\, (\forall s)\ y \preceq f(s) \text{ and } (\exists s)\ y = f(s)\,\}$. Since f is a simple act, $Y_{\min f}$ is nonempty when $\succeq$ is a weak order. We henceforth denote by $y_{\min f}$ an arbitrary element of $Y_{\min f}$. Also, $Y_{\max f}$ and $y_{\max f}$ are defined symmetrically.

Given $f, g \in L_0$ and $\lambda \in [0,1]$, a compound lottery act $\lambda f+(1-\lambda)g$ or $f_\lambda g$ is defined by $(\forall s)\ (\lambda f+(1-\lambda)g)(s) = \lambda f(s)+(1-\lambda)g(s)$. By this operation, L_0 turns out to be a mixture space. A special case of the compound lottery act will turn to be important. We often consider $\lambda y_{\min f} + (1-\lambda) y_{\max f}$, that is, a compound act of the worst-limit act with probability λ and the best-limit act with probability $1-\lambda$.

Axioms and Main Results

In addition to Schmeidler's (1989) well-known axioms[9]: (i) weak order; (ii) comonotonic independence; (iv) continuity; (v) monotonicity and (vii) non-degeneracy, we consider the following axioms which may be imposed on a binary relation $\succeq$ defined on L_0. In the axioms, f and g denote arbitrary elements in L_0 and λ denotes an arbitrary real number such that $\lambda \in (0,1]$.

The first axiom requires that any simple lottery act f dominates some compound lottery act of its worst-limit and best-limit constant acts. In the axiom, ε is a real number such that $\varepsilon \in [0,1)$. The axiom requires that the given relation should hold with respect to this ε. Therefore, whether the axiom is satisfied or not depends on ε, and hence, it is labeled (viii-ε), rather than (viii).

(viii-ε) (ε-Dominance)

$$(\forall f \in L_0)\ \ f \succeq (1-\varepsilon) y_{\min f} + \varepsilon y_{\max f}.$$

Under (i), (ii), (iv), (v), (vii) and (viii-ε), it can be shown (see Lemma

[8] Note that a binary relation $\succeq$ is a weak order if and only if $\succ$ is asymmetric and negatively transitive, where a binary relation $\succ$ is *asymmetric* if $(\forall f, g \in L_0)\ f \succ g \Rightarrow g \not\succ f$ and it is *negatively transitive* if $(\forall f, g, h \in L_0)\ [f \not\succ g \text{ and } g \not\succ h] \Rightarrow f \not\succ h$.

[9] For explicit statements of these axioms, the readers are referred to Schmeidler (1989).

1 of Appendix) that all $f \in L_0$ has the following ε-exuberance equivalence:

$$(\forall f \in L_0)(\exists y_f \in L_c) \;\; f \sim (1-\varepsilon)y_f + \varepsilon y_{\max f}\,, \qquad (13.6)$$

where ε is the one with which (viii-ε) holds. This property shows that all simple lottery acts have their own equivalent compound act consisting of its best-limit constant act with probability ε and some constant act y_f with probability $1-\varepsilon$. Clearly, y_f defined in (13.6) is one way of representing f. We hereafter call it *f's equivalent constant act in ε-exuberance equivalence*.

The next axiom concerns ordering among these equivalent constant acts in ε-exuberance equivalence. In the axiom, ε is a real number such that $\varepsilon \in [0, 1)$. By the same reason given for (viii-ε), we label it (ix-ε), rather than (ix).

(ix-ε) (Best-limit irrelevance) Both of the following hold:

(ix-ε-1) (Affine irrelevance) If there exist y_f, y_g, $y_{f\lambda g} \in L_c$ such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\max f}$, $g \sim (1-\varepsilon)y_g + \varepsilon y_{\max g}$ and $f\lambda g \sim (1-\varepsilon)y_{f\lambda g} + \varepsilon y_{\max f\lambda g}$, then $y_{f\lambda g} \sim \lambda y_f + (1-\lambda)y_g$; and

(ix-ε-2) (Monotone irrelevance) If $(\forall s)$ $f(s) \succeq g(s)$ and there exist y_f, $y_g \in L_c$ such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\max f}$ and $g \sim (1-\varepsilon)y_g + \varepsilon y_{\max g}$, then $y_f \succeq y_g$.

Axiom (ix-ε-1) means that if y_f, y_g and $y_{f\lambda g}$ are the equivalent acts of f, g and $f\lambda g$ in ε-exuberance equivalence, respectively, then $y_{f\lambda g} \sim \lambda y_f + (1-\lambda)y_g$, regardless of characteristics of the best-limits $y_{\max f}$, $y_{\max g}$ and $y_{\max f\lambda g}$. Similarly, Axiom (ix-ε-2) implies that if $f(s) \succeq g(s)$ for all s, then $y_f \succeq y_g$, regardless of characteristics of the best-limits $y_{\max f}$ and $y_{\max g}$. These two axioms imply that the best limits are irrelevant in ordering among equivalent constant acts in ε-exuberance equivalence.

Axioms (viii-ε) and (ix-ε) are closely related to the axioms of Anscombe and Aumann (1963), especially their independence axiom (or equivalently, Schmeidler's (iii) independence). In fact they can be considered as a natural extension of the Anscombe-Aumann theory to the case in which the decision-maker has a hope of the best outcome with the possibility of ε all the time. We will turn to this issue in the next section.

As shown in Schmeidler (1989), the five axioms (i), (ii), (iv), (v) and (vii) as a whole characterize the preference which is represented by the Choquet expected utility with respect to some capacity.[10] The main results of this

[10] For a related axiomatization, see Gilboa and Schmeidler (1989).

paper are the following theorem and corollary. The proof is relegated to Section 4.

Theorem 35 *Given any $\varepsilon \in [0,1)$, a binary relation $\succeq$ defined on L_0 satisfies (i), (ii), (iv), (v), (vii), (viii-ε) and (ix-ε) if and only if there exist a unique finitely additive probability measure μ on (S,Σ), an affine function $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation, such that*

$$f \succeq g \;\Leftrightarrow\; (1-\varepsilon)\int_S u(f(s))\,d\mu(s)+\varepsilon \max_{s\in S} u(f(s)) \geq (1-\varepsilon)\int_S u(g(s))\,d\mu(s)+\varepsilon \max_{s\in S} u(g(s))\,.$$

Let $\mathcal{M} = \mathcal{M}(S,\Sigma)$ be the set of finitely additive probability measures (probability charges) on (S,Σ), let $\varepsilon \in [0,1)$, and let $\mu \in \mathcal{M}$. Let us now define *ε-exuberance of* μ, $\{\mu\}^\varepsilon$, which is a subset of $\mathcal{M}$, by $\{\mu\}^\varepsilon = \{\, (1-\varepsilon)\mu + \varepsilon q \mid q \in \mathcal{M}\,\}$.[11] Then, it follows that

$$\begin{aligned}(\forall f \in L_0) \qquad \int_S u(f(s))\,d\{\mu\}^\varepsilon(s) &\equiv \max\left\{ \int_S u(f(s))\,dp(s) \,\middle|\, p \in \{\mu\}^\varepsilon \right\} \\ &= (1-\varepsilon)\int_S u(f(s))\,d\mu(s) + \varepsilon \max_{s\in S} u(f(s))\,.\end{aligned}$$

Therefore, the following corollary is immediate.

Corollary 10 *Given any $\varepsilon \in [0,1)$, a binary relation $\succeq$ defined on L_0 satisfies (i), (ii), (iv), (v), (vii), (viii-ε) and (ix-ε) if and only if there exist a unique finitely additive probability measure μ on (S,Σ), an affine function $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation, such that*

$$f \succeq g \;\Leftrightarrow\; \int_S u(f(s))\,d\{\mu\}^\varepsilon(s) \geq \int_S u(g(s))\,d\{\mu\}^\varepsilon(s)\,.$$

13.6 Pessimistic and Optimistic Behaviors and Asset Prices

[We prove that the effect of higher hopefulness in the sense of larger ε on asset prices is totally opposite depending on the degree of agents' risk aversion.]

[11] The capacity ν corresponding to ε-contamination of μ is given by

$$(\forall A \in \Sigma) \quad \nu(A) = \begin{cases} 0 & \text{if} \quad A = \phi \\ \varepsilon + (1-\varepsilon)\mu(A) & \text{if} \quad A \neq \phi\,. \end{cases}$$